On best proximity points for multivalued Geometric F - contraction mappings

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Abstract

In this paper, we introduce the concept of property UC^{**} and Geometric F - contraction. We establish and prove the existence of best proximity points for multivalued Geometric F - contraction mappings in complete metric spaces.Our results improved and generalizes the results of Konrawut Khammahawong et.al.

Keywords: best proximity point, Geometric contraction, F - contraction, multivalued mapping, metric space
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1 Introduction

Throughout this paper, for metric space (X, d), we denote $C_b(X)$ by the family of all non-empty closed bounded subsets of a metric space (X, d). The

Pompeiu - Hausdorff metric induced by d on $C_b(X)$ is given by

$$H(A,B) = \max\ \{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$$

for every $A, B \in C_b(X)$, where $d(a,B) = inf\{d(a,b) : b \in B\}$ is the distance from a to $B \subseteq X$

Remark 1. *The following properties of the Pompeieu - Hausdroff metric induced by d are well known.*

1. *H is a metric on $C_b(X)$*
2. *If $A, B \in C_b(X)$ and $h > 1$ be given , then for every $a \in A\ \exists\ b \in B \ni d(a,b) \leq hH(A,B)$*

In 1992, Banach Contraction principle was defined by Banach. Let $T : X \to X$ be a self mapping of a complete metric (X,d) such that $d(Tx,Ty) \leq Ld(x,y)$ for each $x, y \in X$, where $0 \leq L < 1$. Then T has a unique fixed point. Further, since Banach's simplicity, usefulness and applications, it has become a very popular tools in solving the existence problems in many branches of mathematical analysis. Several authors improved, extended and generalized banach's fixed point theorem in many directions.

In a different way, if T is a non-self mapping then there is no fixed point from equation $Tx = x$. The investigation of this case that there is an element x such that $d(x,Tx)$ is minimum. This point becomes a concept of best proximity point theorem, so theorem guarantees the existence of an element x such that $d(x,tx) = d(A,B) = inf\{d(x,y) : x \in A$ and $y \in B\}$ then x is called a best proximity point of non-self mapping T. Since a non-self mapping T has no fixed point, but this mapping gives a best proximity point so it is optimal approximate solution of the fixed point equation $Tx = x$. If $d(A,B) = 0$, then a fixed point and a best proximity point are same point. A best proximity point is reduced to a fixed point if T is a self mapping.

In 1969, Fan [1] be the first who study in area of the best proximity point theorem. He established a classical best approximation theorem. Afterwards several researchers have been extended the best proximity theorem in many directions.

In the same year, Nadler [3] gives new idea of the Banach contraction principle. Researcher extended the theorem from single valued mapping to multivalued mapping.

The purpose of this article is to first introduce the notion of UC^{**} property and geometric F - contraction pair. Moreover, we apply this results in uniformly convex Banach space. We also study some results and give illustrative example of our theorem. Nadler [3] also combine the idea of Lipschitz mappings with multivalued mappings and fixed point theorems as follows. Recently Wardowski [5] proved one of interesting in fixed point theorem which is F - contraction mapping on complete metric spaces.

The aim of this paper, we introduce the notation and concept of multivalued Geometric F- contraction pair and prove a best proximity point such a mappings in a complete metric space via property UC^{**}.

2 Preliminaries

In this paper, we give some basic definitions and concepts related to the main results of this article. Throughout this paper we denote $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ by the set of positive integers, the set of real numbers and the set of non- negative real numbers respectively.

Definition 1. *[10] Let A and B be non-empty subsets of a metric spaces X and $T : A \to 2^B$ be a multivalued mapping. A point $x \in A$ is said to be best proximity point of a multivalued mapping T if it satisfies the following condition*

$$d(x, Tx) = d(A, B)$$

We have that a best proximity point reduces to a fixed point for a multivalued mapping if the underlying mapping is a self mapping.

Definition 2. *[10] A Banach space $(X, \| . \|)$ is said to be*

1. *strictly convex if the following condition holds for all $x, y \in X$:*

$$\| x \| = 1 \text{ and } x \neq y \Rightarrow \| \frac{x+y}{2} \| < 1$$

2. *uniformly convex if for each ϵ with $0 < \epsilon \leq 2$, there exists $\delta > 0$ such that the following condition holds for all $x, y \in X$:*

$$\| x \| \leq 1, \| y \| \leq 1 \text{ and } \| x - y \| \geq \epsilon \Rightarrow \| \frac{x+y}{2} \| < 1 - \delta$$

Remark 2. *It is easy to see that a uniformly convexity implies strictly convexity but the converse is not true.*

Definition 3. *[7] Let A and B be nonempty subsets of a metric space X. The ordered pair (A, B) is said to satisfy the property UC if the following holds:*

If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ be a sequence in B such that $d(x_n, y_n) \to d(A, B)$ and $d(z_n, y_n) \to d(A, B)$ then $d(x_n, z_n) \to 0$

Example 1. *[7] The following are some examples of a pair of nonempty subsets (A, B) satisfying the property UC*

1. *Every pair of nonempty subsets A, B of a metric space(X, d) such that $d(A, B) = 0$*
2. *Every pair of nonempty subset A, B of a uniformly convex Banach space X such that A is convex.*
3. *Every pair of nonempty subset A, B of a strictly convex Banach space where A is convex and relatively compact and the closure of B is weakly compact.*

Definition 4. *[6] Let A and B be nonempty subsets of a metric space (X, d). The ordered pair (A, B)satisfies the property UC^* if (A, B) has property UC and the following conditions holds: If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ be a sequence in B satisfying*

1. *$d(z_n, y_n) \to d(A, B)$ as $n \to \infty$*
2. *For each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$d(x_m, y_n) \leq d(A, B) + \epsilon$$

for all $m > n \geq N$

then $d(x_n, z_n) \to 0$ as $n \to \infty$.

Example 2. *The following are some examples of a pair of nonempty subsets (A, B) satisfying the property UC^**

1. *Every pair of nonempty subsets A and B of a metric space(X, d) such that $d(A, B) = 0$*
2. *Every pair of nonempty closed subset A and B of a uniformly convex Banach space X such that A is convex.(see Lemma 3.7 in [4])*

Wardowski [5] defined the following contraction which was called F- contraction as follows:

Definition 5. *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping which is satisfying the following conditions:*

(F1) F is strictly increasing i.e., for all $\alpha, \beta \in \mathbb{R}^+$, $F(\alpha) < F(\beta)$ whenever $\alpha < \beta$.

(F2) For each sequence $\{\alpha_n\}_{n\in N}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ iff $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k(\alpha) = 0$

We denote by $\mathcal{F}$ the family of all functions F that satisfy the conditions (F1)- (F3).For examples of the function F the reader is referred to [5] and [8]

Definition 6. *Let (X, d) be a metric space. A self-mapping T on X is called an F- contraction mapping if there exists $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\forall x, y \in X, [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))] \qquad (1)$$

Remark 3. *From $(F1)$ and (1) it is easy to see that every F- contraction is necessarily continuous.*

Definition 7. *Let A and B be nonempty subsets of a metric space X.*
A map $T : A \cup B \to A \cup B$ is a geometric contraction map if
(i) $T(A) \subset B$ and $T(B) \subset A$.
(ii) For some $\alpha \in (0, 1)$ and all $x \in A$ and $y \in B$ we have

$$d(Tx, Ty) \leq d(x, y)^\alpha d(A, B)^{1-\alpha}$$

Example 3. *If $A = \{(x, 0) : x \geq 1\}$, $B = \{(0, y) : y \geq 1\}$, and $T(x, y) = (\sqrt{y}, \sqrt{x})$. Then $d(A, B) = \sqrt{2}$ and $\alpha = \frac{1}{2}$,*

$$\begin{aligned} \| T(x,0) - T(0,y) \| &= \| (0, \sqrt{x}) - (\sqrt{y}, 0) \| \\ &= \| (\sqrt{y}, \sqrt{x}) \| \\ &= \sqrt{x + y} \\ &\leq \sqrt{\sqrt{2}\sqrt{x^2 + y^2}} \\ &= \sqrt{d(A, B) \| (x, 0) - (0, y) \|} \end{aligned}$$

Hence T is a geometric contraction map with respect to $\alpha = \frac{1}{2}$.

3 Main Results

In this section, we introduce the notion of property UC^{**} and concept of multivalued geometric F - contraction pair and prove a best proximity point such a mappings in a complete metric space.

Definition 8. *Let A and B be nonempty subsets of a metric space (X, d). The ordered pair (A, B)satisfies the property UC^{**} if (A, B) has property UC and the following conditions holds:*
If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ be a sequence in B satisfying

1. *$d(z_n, y_n) \to d(A, B)$ as $n \to \infty$*
2. *For each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$d(x_m, y_n) \leq \epsilon d(A, B)$$

for all $m > n \geq N$

then $d(x_n, z_n) \to 0$ as $n \to \infty$.

Definition 9. *Let A and B be non-empty subsets of a metric space. Let $T : A \to 2^B$ and $S : B \to 2^A$ be multivalued mappings. The ordered pair (T, S) is said to be a multivalued Geometric F - contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that*
$H(Tx, Sy) > 0 \implies 2T + F(H(Tx, Sy)) \leq F(d(x, y)^{\alpha} dist(A, B)^{1-\alpha})$,
For all $x, y \in X$, where $\alpha \in (0, 1)$.

Theorem 1. *Let A and B be non-empty closed subsets of a complete metric space X such that (A, B) and (B, A) satisfy the property UC^{**}.*
Let $T : A \to C_b(A)$ and $S : B \to C_b(A)$. If (T, S) is a multivalued geometric F - contraction pair, then T has a best proximity point in A (or) S has a best proximity point in B.

Proof. We divide the case into two.
Case 1: Assume that $d(A, B) = 0$

Now, we will construct the sequence $\{x_n\}$ in X as follows. Let $x_0 \in A$ be an arbitrary point. Since $Tx_0 \in C_b(B)$, we can choose $x_1 \in Tx_0$. If $Tx_0 \neq Sx_1$, since F is continuous from the right then there exist a real number $h > 1$ and $\tau > 0$ such that

$$F(hH(Tx_0, Sx_1)) < F(H(Tx_0, Sx_1)) + \tau$$

from $d(x_1, Sx_1) < hH(Tx_0, Sx_1)$, we deduce that there exists $x_2 \in Sx_1$ such that

$$d(x_1, x_2) \leq hH(Tx_0, Sx_1)$$

It follows from the definition of F, we have

$$\begin{aligned} F(d(x_1, x_2)) &\leq F(hH(Tx_0, Sx_1)) \\ &< F(H(Tx_0, Sx_1)) + \tau \end{aligned}$$

which implies

$$\begin{aligned} F(d(x_1, x_2)) &\leq F(H(Tx_0, Sx_1)) + \tau \\ &\leq F(d(x_0, x_1)^{\alpha}) + \tau - 2\tau \\ &\leq F(d(x_0, x_1)^{\alpha}) - \tau \\ &\leq F(d(x_0, x_1)) - \tau \end{aligned}$$

otherwise, if $Tx_2 \neq Sx_1$, since F is continuous from the right then there exists a real number $h > 1$ and $\tau > 0$ such that

$$F(hH(Sx_1, Tx_2)) < F(H(Sx_1, Tx_2)) + \tau$$

Now from $d(x_2, Tx_2) < hH(Sx_1, Tx_2)$, we obtain that there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq hH(Sx_1, Tx_2)$$

Consequently, we get

$$\begin{aligned} F(d(x_2, x_3) &\leq F(hH(Sx_1, Tx_2)) \\ &< F(H(Sx_1, Tx_2)) + \tau \end{aligned}$$

which implies

$$\begin{aligned} F(d(x_2, x_3) &\leq F(H(Sx_1, Tx_2)) + \tau \\ &\leq F(d(x_1, x_2)^{\alpha}) + \tau - 2\tau \\ &\leq F(d(x_1, x_2)^{\alpha}) - \tau \\ &\leq F(d(x_1, x_2)) - \tau \end{aligned}$$

By induction, we can find $\{x_n\}$ such that

$$\begin{aligned} F(d(x_n, x_{n+1}) &\leq F(d(x_{n-1}, x_n)^{\alpha}) - \tau \\ &\leq F(d(x_{n-1}, x_n)^{\alpha}) - \tau \\ &\vdots \\ &\leq F(d(x_0, x_1)^{\alpha}) - n\tau \\ &\leq F(d(x_1, x_2)) - n\tau \end{aligned}$$

Let $\beta_n = d(x_n, x_{n+1})$. From above, we receive $\lim_{n\to\infty} F(\beta_n) = -\infty$ that together with $(F2)$ gives

$$\lim_{n\to\infty} \beta_n = 0$$

Also from $(F3)$ we have there exists $l \in (0,1)$ such that

$$\lim_{n\to\infty} \beta_n^l F(\beta_n) = 0$$

Now, it follows that

$$\begin{aligned} \beta_n^l F(\beta_n) - \beta_n^l F(\beta_0) &\leq \beta_n^l (F(\beta_0) - n\tau) - \beta_n^l F(\beta_0) \\ &\leq \beta_n^l F(\beta_0) - \beta_n^l n\tau - \beta_n^l F(\beta_0) \\ &\leq -\beta_n^l n\tau \\ &\leq 0, for \quad all \quad n \in \mathbb{N} \end{aligned}$$

Letting n as $n \to \infty$, so we obtain

$$n\beta_n^l = 0 \text{ for all } n \in \mathbb{N}$$

From above, $\lim_{n\to\infty} n\beta_n^l = 0$ there exist $n_1 \in \mathbb{N}$ such that $n\beta_n^l \leq 1$ for all $n \geq n_1$. Therefore, $\beta_n \leq \dfrac{1}{n^{\frac{1}{l}}}$, for all $n \geq n_1$.

Let $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. We compute that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \beta_n + \beta_{n+1} + \cdots + \beta_{m-1} \\ &= \sum_{i=1}^{m-1} \beta_i \\ &\leq \sum_{i=n}^{\infty} \beta_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{l}}} \end{aligned}$$

By the convergence of the P series $\sum\limits_{i=n}^{\infty} \dfrac{1}{i^{\frac{1}{l}}}$, so as $n \to \infty$, we obtain $d(x_n, x_m) \to 0$ as $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since completeness of X, then $\{x_n\}$ converges to some point $Z \in X$. Clearly, the subsequence $\{x_{2n}\}$ and $\{x_{2n-1}\}$ converge to same point z. Since A and B are called, we obtain that $Z \in A \cap B$.

From 8, for all $x, y \in X$ and $\alpha \in (0,1)$ with $H(Tx, Sy) > 0$ and $d(A, B) = 0$, we get

$$2\tau + F(H(Tx, Sy)) \leq F(d(x, y)^{\alpha}) \leq F(d(x, y))$$

Since F is strictly increasing, we get $H(Tx, Sy) < d(x, y)$ and so $H(Tx, Sy) \leq d(x, y)$ for all $x, y \in X$. Then

$$d(x_{2n+1}, Tz) \leq H(Sx_{2n}, Tz) \leq d(x_{2n}, z)$$

passing to limit $n \to \infty$, we obtain

$$d(z, Tz) = d(A, B)$$

Similarly, we also derive $d(Sz, z) = d(A, B)$.

Case 2: We will show that T or S have best proximity points in A and B respectively. Under the assumption of $d(A, B) > 0$, suppose to the contrary, that is for all $a \in A$, $d(a, Ta) > d(A, B)$ and for all $b \in B$, $d(sb^{'}, b^{'}) > d(A, B)$. For each $a \in A$ and $b \in Ta$, we have

$$d(A, B) < d(a, Ta) \leq d(a, b) \tag{2}$$

since (T, S) is a multivalued geometric F - contraction pair, such that

$$\begin{aligned} F(H(Ta, Sb)) &\leq F(d(a, b)^{\alpha} d(A, B)^{1-\alpha}) - 2\tau \\ &< F(d(a, b)^{\alpha} d(A, B)^{1-\alpha}) \end{aligned} \tag{3}$$

for all $a \in A$ and $b \in Ta$. Since F is strictly increasing, we get

$$H(Ta, Sb) < d(a, b)^{\alpha} d(A, B)^{1-\alpha} \tag{4}$$

for all $a \in A$ and $b \in Ta$.
similarly, we have that for each $b^{'} \in B$ and $a^{'} \in Sb^{'}$, we get

$$F(H(Ta^{'}, Sb^{'})) < F(d(a^{'}, b^{'})^{\alpha} d(A, B)^{1-\alpha}) \tag{5}$$

and

$$H(Ta^{'}, Sb^{'})) < (d(a^{'}, b^{'})^{\alpha} d(A, B)^{1-\alpha}) \tag{6}$$

Next we will construct the sequence $\{x_n\}$ in $A \cup B$. Let $\{x_0\}$ be arbitrary point of A and $x_1 \in Tx_0 \subseteq B$.
From (3), there exist $x_2 \in Sx_1$, such that

$$\begin{aligned} F(d(x_1, x_2)) &\leq F(H(Tx_0, Sx_1)) + \tau \\ &\leq F(d(x_0, x_1)^{\alpha} d(A, B)^{1-\alpha}) - 2\tau + \tau \\ &\leq F(d(x_0, x_1)^{\alpha} d(A, B)^{1-\alpha}) - \tau \\ &< F(d(x_0, x_1)^{\alpha} d(A, B)^{1-\alpha}) \end{aligned}$$

and since F is strictly increasing, we get

$$d(x_1, x_2) < d(x_0, x_1)^{\alpha} d(A, B)^{1-\alpha} \tag{7}$$

Since $x_1 \in B$ and $x_2 \in Sx_1$ from (5) we can find $x_3 \in Tx_2$ such that

$$d(x_2, x_3) < d(x_1, x_2)^{\alpha} d(A, B)^{1-\alpha}) \tag{8}$$

Consequently, we define the sequence x_n in $A \cup B$ such that

$$x_{2n-1} \in Tx_{2n-2},\ x_{2n} \in Sx_{2n-1}$$

and

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n)^{\alpha} d(A, B)^{1-\alpha} \tag{9}$$

for all $n \in \mathbb{N}$. since $d(A, B) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d(x_0, x_{n+1}) &< d(x_{n-1}, x_n)^{\alpha} d(A, B)^{1-\alpha}) \\ &\leq d(x_{n-1}, x_n)^{\alpha} d(x_{n-1}, x_n)^{1-\alpha} \\ &\leq d(x_{n-1}, x_n)^{\alpha} d(x_{n-1}, x_n) d(x_{n-1}, x_n)^{-\alpha} \\ &\leq d(x_{n-1}, x_n) \end{aligned} \tag{10}$$

and

$$\begin{aligned} d(x_n, x_{n+1}) &< d(x_{n-1}, x_n)^{\alpha} d(A, B)^{1-\alpha} \\ &< (d(x_{n-2}, x_{n-1})^{\alpha} d(A, B)^{1-\alpha})^{\alpha} d(A, B)^{1-\alpha} \\ &< d(x_{n-2}, x_{n-1})^{\alpha^2} d(A, B)^{\alpha(1-\alpha)} d(A, B)^{1-\alpha} \\ &< d(x_{n-2}, x_{n-1})^{\alpha^2} d(A, B)^{\alpha-\alpha^2+1-\alpha} \\ &< d(x_{n-2}, x_{n-1})^{\alpha^2} d(A, B)^{1-\alpha^2} \\ &\vdots \\ &< d(x_0, x_1)^{\alpha^n} d(A, B)^{1-\alpha^n} \end{aligned} \tag{11}$$

Hence $d(A, B) \leq d(x_n, x_{n+1}) < d(x_0, x_1)^{\alpha^n} d(A, B)^{1-\alpha^n}$ for all $n \in \mathbb{N}$.
Since $\alpha \in (0, 1)$ we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B) \tag{12}$$

from (12) we get

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = d(A, B) \tag{13}$$

and

$$\lim_{n\to\infty} d(x_{2n+2}, x_{2n+1}) = d(A, B) \tag{14}$$

Since $\{x_{2n}\}$ and $\{x_{2n+2}\}$ are two sequences in A and x_{2n+1} in sequence B with (A, B) which satisfies UC^{**}, we derive that

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = 0 \tag{15}$$

Since (B,A) satisfies the property UC^{**} and by (12), we have

$$\lim_{n\to\infty} d(x_{2n-1}, x_{2n+1}) = 0 \tag{16}$$

Next, we will show that for each $\epsilon > 0$, there exists $N \in \mathbb{N} \ni$ for all $m > n \geq N$, we have

$$\lim_{n\to\infty} d(x_{2m}, x_{2n+1}) \leq \epsilon d(A, B) \tag{17}$$

suppose the contrary that there exists $\epsilon_0 > 0$ such that for each $k \geq 1$ there is $m_k > n_k \geq k$ such that

$$d(x_{2m_k}, x_{2n_{k+1}}) > \epsilon_0 d(A, B) \tag{18}$$

Moreover corresponding to n_k, we can choose m_k in such a way that is the smallest integer with $m_k > n_k \geq k$ satisfying (18). Then we obtain

$$d(x_{2m_k}, x_{2n_{k+1}}) > \epsilon_0 d(A, B) \tag{19}$$

and

$$d(x_{2(m_{k-1})}, x_{2n_{k+1}}) \leq d(A, B) \times \epsilon_0 \tag{20}$$

From (19), (20) and the triangle inequality, we obtain

$$\begin{aligned} \epsilon_0 d(A, B) &< d(x_{2m_k}, x_{2n_{k+1}}) \\ &\leq d(x_{2m_k}, x_{2(m_k-1)}) + d(x_{2(m_k-1)}, x_{2n_k} + 1) \\ &\leq d(x_{2m_k}, x_{2(m_k-1)}) + \epsilon_0 d(A, B) \end{aligned} \tag{21}$$

using the fact that $\lim_{k\to\infty} d(x_{2mk}, x_{2(m_k-1)}) = 0$. Letting $k \to \infty$ in (21), we get

$$\lim_{k\to\infty} d(x_{2m_k}, x_{2n_{k+1}}) = \epsilon_0 d(A, B) \tag{22}$$

From (9) and (10) and (T, S) is a multivalued geometric F-contraction pair, we obtain

$$\begin{aligned} d(x_{2m_k}, x_{2n_k+1}) &\leq d(x_{2m_k}, x_{2m_{k+2}}) + d(x_{2m_{k+2}}, x_{2n_{k+3}}) + d(x_{2n_{k+3}}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2m_{k+2}}) + d(x_{2m_{k+1}}, x_{2n_{k+}}) + d(x_{2n_{k+3}}, x_{2n_k+1}) \\ &< d(x_{2m_k}, x_{2m_{k+2}}) + d(x_{2n_{k+3}}, x_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1})^{\alpha} d(A, B)^{1-\alpha} \end{aligned} \tag{23}$$

Letting $k \to \infty$ in (23) and using (15), (16) and (23), we have

$$\begin{aligned}\epsilon_0 d(a,b) &< (d(A,B) \times \epsilon_0)^{\alpha} d(A,B)^{1-\alpha} \\ &< d(A,B)^{\alpha} \times \epsilon_0^{\alpha} d(A,B) d(A,B)^{-\alpha} \\ &< \epsilon_0^{\alpha} d(A,B)\end{aligned}$$

which is a contradiction. Therefore (17) holds. Since (13) and (17) hold, by using property UC^{**} of (A,B) we obtain $d(x_{2n}, x_{2m}) \to 0$ as $n \to \infty$.
$\therefore \{x_{2n}\}$ is a Cauchy sequence. Since X is complete and A is closed, we have

$$\lim_{n\to\infty} x_{2n} = p \tag{24}$$

for some $p \in \overline{A} = A$. But

$$\begin{aligned}d(A,B) &\le d(p, x_{2n-1}) \\ &\le d(p, x_{2n}) + d(x_{2n}, x_{2n-1})\end{aligned}$$

for all $n \in \mathbb{N}$. From (12) and (24)

$$\lim_{n\to\infty} d(p, x_{2n-1}) = d(A,B) \tag{25}$$

Since

$$\begin{aligned}d(A,B) &< d(x_{2n}, T_p) \\ &< H(S_{2n-1}, T_p) \\ &= H(T_p, S_{2n-1}) \\ &\le d(p, x_{2n-1})^{\alpha} d(A,B)^{1-\alpha} \\ &\le d(p, x_{2n-1})\end{aligned} \tag{26}$$

for all $n \in \mathbb{N}$. From (24) and (25)

$$d(p, T_p) = d(A,B) \tag{27}$$

In a similar mode, we can conclude that the sequence $\{x_{2n-1}\}$ is a cauchy sequence in B. Since X is complete and B is closed, we obtain

$$\lim_{n\to\infty} x_{2n-1} = q \tag{28}$$

for some $q \in \overline{B} = B$. Since

$$\begin{aligned}d(A,B) &\le d(x_{2n}, q) \\ &\le d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, q)\end{aligned}$$

for all $n \in \mathbb{N}$. It follows (12) and (28) that

$$\lim_{n\to\infty} d(x_{2n}, q) = d(A, B) \tag{29}$$

Since

$$\begin{aligned} d(A,B) &< d(S_q, x_{2n+1}) \\ &\leq H(S_q, Tx_{2n}) \\ &= H(Tx_{2n}, S_q) \\ &< d(x_{2n}, q)^{\alpha} d(A,B)^{1-\alpha} \\ &\leq d(x_{2n}, q) \end{aligned} \tag{30}$$

for all $n \in \mathbb{N}$, then by (28) and (29), we have

$$d(q, S_q) = d(A, B) \tag{31}$$

from (27) and (31), we get a contradiction.
$\therefore$ T has a best proximity point in A or S has a best proximity point in B. This completes the proof. ∎

4 References

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